Math 250

7.2-1st The Natural Logarithmic Function: Differentiation

Objectives

1) Use and understand properties of $f(x) = \ln x$.

2) Understand the definition of the number e

3) Find derivatives of functions involving the natural log function.

☆ 4) Use logarithmic differentiation to find simpler-looking derivatives of complicated functions

Math 250 [redacted]

[redacted]

Recall: in Math 70 or 101/244, we had the question of how to solve exponential equations with unlike bases, e.g.

$$2^x = 3.$$

The solution to this conundrum was to invent the logarithm to be a function which is the inverse of an exponential – a function having needed properties.

Now: We have two questions

#1: How do we integrate $\int x^{-1}\,dx = \int \frac{dx}{x}$?

#2: Whence the number e ?

The solutions to these questions are inventions also –
In #1; a function having the needed properties.
In #2; a number having the needed properties.

Definitions

Soln #1 $\ln x = \int_1^x \frac{1}{t}\,dt \quad x > 0.$ ln(x) is the function that allows us to integrate

Soln #2 $\ln e = \int_1^e \frac{1}{t}\,dt = 1$ e is the positive real number that makes this true

We won't spend time on why

$\ln x = \int_1^x \frac{1}{t}\,dt$ is the same as $\ln x = \log_e x$, with $e \approx 2.71828182846\ldots$ irrational

[See notes online for this.]

Math 250

If $\ln x = \int_1^x \frac{1}{t}\,dt$ by definition

then $\frac{d}{dx}\left[\ln x\right] = \frac{d}{dx}\left[\int_1^x \frac{1}{t}\,dt\right]$ taking derivatives both sides

But 2nd FTC tells us this means:

$$* \;\; \boxed{\frac{d}{dx}\left[\ln x\right] = \frac{1}{x} \quad x>0} \;\; *$$

Memorize this result!

$$\boxed{\text{and } \frac{d}{dx}\left[\ln|x|\right] = \frac{1}{x} \quad x \neq 0}$$

← by a symmetry argument.

The chain rule means:

$$\frac{d}{dx}\left[\ln|g(x)|\right] = \underbrace{\frac{1}{g(x)}}_{\text{outside function}} \cdot \underbrace{g'(x)}_{\text{inside function}}$$

However: notice that $|g(x)|$ is a requirement! Otherwise ln(neg) is not defined.

Similarly $g(x) \neq 0$.

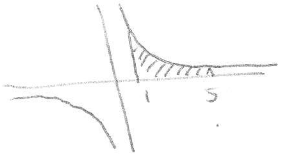
So, if $\frac{d}{dx} \ln x = \frac{1}{x}$

then $\int \frac{1}{x}\,dx = \ln x + C$, Right? WRONG.

what's the domain of $f(x) = \frac{1}{x}$? $(-\infty, 0) \cup (0, \infty)$
all $\mathbb{R}$ except $x = 0$.

what's the domain of $g(x) = \ln x$? $(0, \infty)$ only
No logs of negatives!

✓ ① Consider $\int_1^5 \frac{1}{x}\,dx$



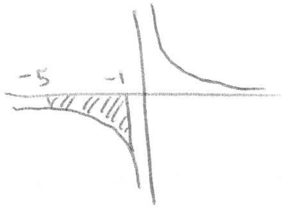
on $[1,5]$ $x > 0$ so both $\frac{1}{x}$ and $\ln x$ are defined.

$= \ln x \Big|_1^5$

$= \ln 5 - \ln 1$

$= \boxed{\ln 5} \approx 1.6094$ check fnInt(1/x, x, 1, 5)
MATH 9. ≈ 1.6094

But what about $\int_{-5}^{-1} \frac{1}{x}\,dx$
✓ ②



on $[-5, -1]$ $x < 0$ so $\frac{1}{x}$ is defined but $\ln x$ is not

But since $f(x) = \frac{1}{x}$ is symmetric about the origin, we suspect that this integral should equal $-\ln 5 \approx -1.6094$.

So we use absolute values! check fnInt(1/x, x, -5, -1)
≈ -1.6094

$$\int_{-5}^{-1} \frac{1}{x}\,dx = \ln|x| \Big|_{-5}^{-1} = \ln|-1| - \ln|-5| = \ln 1 - \ln 5 = \boxed{-\ln 5}$$

THUS

$$\boxed{\int \frac{1}{x}\,dx = \ln|x| + C}$$

* Leaving off absolute values is WRONG *

Math 250

Find derivatives

X ① $f(x) = \ln(x-1)$ $\quad x>1$

$$\boxed{f'(x) = \frac{1}{x-1}}$$

X ② $h(x) = \ln(2x^2+1)$

$$h'(x) = \frac{1}{2x^2+1} \cdot 4x = \boxed{\frac{4x}{2x^2+1}}$$

chain rule

✓ ③ $y = x^2 \ln x$ $\quad x>0$

product rule!

$$y' = x^2 \cdot \frac{1}{x} + \ln x \cdot 2x$$

$$= \boxed{x + 2x \ln x}$$

✓ ④ $y = \ln\sqrt{x^2-4}$ $\quad x>2$

$$y = \ln(x^2-4)^{1/2}$$

$$y' = \underbrace{\frac{1}{(x^2-4)^{1/2}}}_{\text{deriv of natural log}} \cdot \underbrace{\tfrac{1}{2}(x^2-4)^{-1/2}}_{\text{deriv of square root}} \cdot \underbrace{2x}_{\text{deriv of inside of sq root.}}$$

← chain rule twice

$$y'(x) = \frac{1 \cdot 2x}{2\sqrt{x^2-4}\,\sqrt{x^2-4}}$$

$$\boxed{y' = \frac{x}{x^2-4}}$$

Method 2

OR Use log properties!

$$y = \tfrac{1}{2}\ln(x-2) + \tfrac{1}{2}\ln(x+2)$$

$$y'(x) = \frac{1}{2} \cdot \frac{1}{x-2} + \frac{1}{2} \cdot \frac{1}{x+2}$$

$$= \frac{x+2 + x-2}{2(x+2)(x-2)}$$

$$= \frac{2x}{x(x^2-4)} = \frac{x}{x^2-4}$$

Note: It is not unusual for log derivatives to simplify unexpectedly.

Math 250 [redacted]

Find equation of tangent line to graph at given point.

✗ ⑤ $f(x) = 4 - x^2 - \ln(\frac{1}{2}x + 1)$ at $(0,4)$

check $f(0) = 4 - 0^2 - \ln(\frac{1}{2} \cdot 0 + 1)$
$= 4 - \ln(1) = 4$ ✓

$$f'(x) = -2x - \frac{1}{\frac{1}{2}x+1} \cdot \frac{1}{2}$$

$$= -2x - \frac{\cancel{2}}{x+2} \cdot \frac{1}{\cancel{2}}$$

$$= -2x - \frac{1}{x+2}$$

$$f'(0) = -2(0) - \frac{1}{0+2}$$

$$= 0 - \frac{1}{2}$$

$= -\frac{1}{2} =$ slope of tangent line at $(0,4)$

$y - y_1 = m(x - x_1)$ point-slope form

$y - 4 = -\frac{1}{2}(x - 0)$

$y - 4 = -\frac{1}{2}x$

$2y - 8 = -x$

$\boxed{x + 2y - 8 = 0}$

Use implicit differentiation to find derivative.

✓ ⑥ $\ln xy + 5x = 30$ use log properties before diff.

$\ln x + \ln y + 5x = 30$

$\frac{1}{x} + \frac{1}{y}\frac{dy}{dx} + 5 = 0$ differentiate all terms, using chain rule $\frac{dy}{dx}$ for derivatives of y.

$\frac{1}{y}\frac{dy}{dx} = -\frac{1}{x} - 5$ isolate $\frac{dy}{dx}$

$\boxed{\frac{dy}{dx} = -y\left[\frac{1}{x} + 5\right]}$

OR $\boxed{\frac{dy}{dx} = -\frac{y}{x} - 5y}$

Math 250

Logarithmic Differentiation

- Perhaps better if called Natural Logarithmic Differentiation
- Useful when taking derivatives of functions which have many products, quotients or exponents (including radicals).
- Essential when taking derivatives of exponentials bases $\neq e$.

step 1: Take ln both sides of equation

* CAUTION * One ln of entire LHS
One ln of entire RHS

step 2: Expand using log properties

$$\ln(a^k) = k\ln a$$
$$\ln(ab) = \ln a + \ln b$$
$$\ln\left(\frac{a}{b}\right) = \ln a - \ln b$$

} Algebra

step 3: Differentiate using implicit differentiation } Calculus

step 4: Substitute back for y } Algebra

[**step 5:** Recombine fractions if requested.] } Algebra

Use log. differentiation to find $\frac{dy}{dx}$

✓ ⑦ $$y = \sqrt{\frac{x^2-1}{x^2+1}}$$

$$\ln y = \ln\sqrt{\frac{x^2-1}{x^2+1}} = \ln\left(\frac{(x+1)(x-1)}{(x^2+1)}\right)^{1/2}$$

$$\ln y = \frac{1}{2}\ln(x+1) + \frac{1}{2}\ln(x-1) - \frac{1}{2}\ln(x^2+1)$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{2}\cdot\frac{1}{x+1} + \frac{1}{2}\cdot\frac{1}{x-1} - \frac{1}{2}\cdot\frac{1}{x^2+1}\cdot 2x$$

$$\frac{dy}{dx} = y\left[\frac{1}{2(x+1)} + \frac{1}{2(x-1)} - \frac{x}{x^2+1}\right]$$

$$\boxed{\frac{dy}{dx} = \sqrt{\frac{x^2-1}{x^2+1}}\left[\frac{1}{2(x+1)} + \frac{1}{2(x-1)} - \frac{x}{x^2+1}\right]}$$

To recombine, find LCD $= 2(x+1)(x-1)(x^2+1)$

heinous algebra not required by Anton book
see next page for this solution

cont

$$\frac{dy}{dx} = \frac{\sqrt{x+1}\,\sqrt{x-1}}{\sqrt{x^2+1}}\left[\frac{(x-1)(x^2+1)+(x+1)(x^2+1)-x(x+1)(x-1)\cdot 2}{2(x+1)(x-1)(x^2+1)}\right]$$

$$\frac{dy}{dx} = \frac{1}{2}\cdot\frac{\sqrt{x+1}}{x+1}\cdot\frac{\sqrt{x-1}}{x-1}\cdot\frac{1}{\sqrt{x^2+1}\cdot(x^2+1)}\left[x^3-x^2+x-1+x^3+x^2+x+1-2x\cdot(x^2-1)\right]$$

$$\frac{dy}{dx} = \frac{1}{2}\cdot\frac{1}{\sqrt{x+1}}\cdot\frac{1}{\sqrt{x-1}}\cdot\frac{1}{(x^2+1)^{3/2}}\left(2x^3+2x-2x^3+2x\right)$$

$$\frac{dy}{dx} = \frac{4x}{2\sqrt{x+1}\,\sqrt{x-1}\,(x^2+1)^{3/2}}$$

$$\boxed{\frac{dy}{dx} = \frac{2x}{\sqrt{x+1}\,\sqrt{x-1}\,(x^2+1)^{3/2}} \quad \text{or} \quad \frac{2x}{\sqrt{(x+1)(x-1)(x^2+1)^3}}}$$

Not b^x or x^b

Can't treat as exponential

OR as polynomial!

Must use ln differentiation

⑧ Differentiate $y = x^x$

Use ln differentiation

$$\ln y = \ln x^x$$

take natural logs both sides

$$\ln y = \underbrace{x\cdot\ln x}_{\text{product rule!}}$$

log property ("power rule")

$$\frac{1}{y}\cdot\frac{dy}{dx} = x\cdot\frac{d}{dx}[\ln x] + \frac{d}{dx}[x]\cdot\ln x$$

Implicit differentiation

$$\frac{1}{y}\cdot\frac{dy}{dx} = x\cdot\frac{1}{x} + 1\cdot\ln x$$

$$\frac{1}{y}\frac{dy}{dx} = 1 + \ln x$$

simplify

$$\frac{dy}{dx} = y(1+\ln x)$$

isolate $\frac{dy}{dx}$

$$\boxed{\frac{dy}{dx} = x^x(1+\ln x)}$$

Note: $f(x) = x^x$ or any function having x as both base and exponent is called a Tower Function.

Math 250

Differentiate.

X ⑨ $y = \sqrt{x^2(x+1)(x+2)}$ $\quad x > 0.$

step 1: $\ln y = \ln \sqrt{x^2(x+1)(x+2)}$

step 2: $\ln y = \frac{1}{2}\left[\ln x^2(x+1)(x+2)\right]$

$$\ln y = \frac{1}{2}\ln x^2 + \frac{1}{2}\ln(x+1) + \frac{1}{2}\ln(x+2)$$

$$\ln y = \ln x + \frac{1}{2}\ln(x+1) + \frac{1}{2}\ln(x+2)$$

step 3: $\frac{1}{y}\cdot\frac{dy}{dx} = \frac{1}{x} + \frac{1}{2(x+1)} + \frac{1}{2(x+2)}$

$$\frac{dy}{dx} = y\left(\frac{1}{x} + \frac{1}{2(x+1)} + \frac{1}{2(x+2)}\right)$$

$$\frac{dy}{dx} = y\left(\frac{2(x+1)(x+2) + x(x+2) + x(x+1)}{2x(x+1)(x+2)}\right)$$

$$\frac{dy}{dx} = y\left(\frac{2(x^2+3x+2) + x^2+2x + x^2+x}{2x(x+1)(x+2)}\right)$$

step 4: $\frac{dy}{dx} = \frac{\sqrt{x^2(x+1)(x+2)}}{2x(x+1)(x+2)}\cdot(2x^2+6x+4+2x^2+3x)$

$$\frac{dy}{dx} = \frac{\sqrt{x^2(x+1)(x+2)}\,(4x^2+9x+4)}{2x(x+1)(x+2)}$$

$$\frac{dy}{dx} = \frac{x\sqrt{(x+1)(x+2)}\,(4x^2+9x+4)}{2x\,(x+1)(x+2)}$$

$$\boxed{\frac{dy}{dx} = \frac{4x^2+9x+4}{2\sqrt{(x+1)(x+2)}}}$$

Math 250

Use calculus to show intervals of increasing or decreasing and concave up or down.

⑨ $f(x) = \ln x$

$f'(x) = \frac{1}{x} = x^{-1}$

$f''(x) = -x^{-2} = \frac{-1}{x^2}$

critical values $\frac{1}{x} = 0$ none

$\frac{1}{x}$ undef $x = 0$

$\boxed{\text{increasing } (0, \infty)}$

exclude 0 b/c f(0) undef

$\boxed{\text{decreasing none}}$

b/c f(x) undef $x < 0$

$f''(x) = 0$ none

$f''(x)$ undef $x = 0$

$\boxed{\text{concave up } (0, \infty)}$

exclude $(-\infty, 0)$ b/c f(x) undef $x < 0$.

(10) $f(x) = \frac{\ln x}{x} = x^{-1} \ln x$

a) Find all relative extrema.

b) Find all inflection points.

c) Sketch graph.

a) $f'(x) = x^{-1} \frac{d}{dx}(\ln x) + \frac{d}{dx}(x^{-1}) \cdot \ln x$

$= \frac{1}{x} \cdot \frac{1}{x} + -x^{-2} \ln x$

$= \frac{1}{x^2} - \frac{\ln x}{x^2}$

$= \frac{1 - \ln x}{x^2}$

Math 250

cont.

a) $f'(x) = 0$ $\quad 1 - \ln x = 0$

$1 = \ln x$

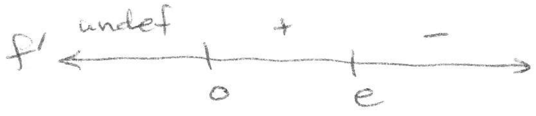
$\ln x = 1$

$e^1 = x$

$x = e$

$f'(x)$ undef $\quad x^2 = 0$

$x = 0$



$f(e) = \frac{\ln e}{e} = \frac{1}{e}$

$\boxed{(e, \tfrac{1}{e}) \text{ relative max}}$

b) $f'(x) = x^{-2}(1 - \ln x)$

$f''(x) = x^{-2} \cdot \frac{d}{dx}(1 - \ln x) + \frac{d}{dx}(x^{-2}) \cdot (1 - \ln x)$

$= x^{-2}\left(-\frac{1}{x}\right) - 2x^{-3}(1 - \ln x)$

$= \frac{-1}{x^3} - \frac{2(1 - \ln x)}{x^3}$

$= \frac{-1 - 2 + 2\ln x}{x^3}$

$= \frac{2\ln x - 3}{x^3}$

$f''(x) = 0$ $\quad 2\ln x - 3 = 0$

$2\ln x = 3$

$\ln x = \frac{3}{2}$

$e^{3/2} = x$

$f''(x)$ undef $\quad x^3 = 0 \quad x = 0$

$f(e^{3/2}) = \frac{\ln(e^{3/2})}{e^{3/2}} = \frac{3}{2}e^{-3/2}$

$\boxed{(e^{3/2}, \tfrac{3}{2}e^{-3/2}) \text{ inflection point}}$

c) $f(x) = \frac{\ln x}{x}$

x-int: $\quad 0 = \ln x \quad x = 1 \quad (1, 0)$

y-int: $\quad \frac{\ln(0)}{0}$ undef $\quad$ none

Math 250

con't

c) symmetry: $f(-x) = \frac{\ln(-x)}{(-x)}$ if $x > 0$ $f(-x)$ not defined.

no symmetry WRT x, y, or origin.

vertical asymptotes: $\underline{x = 0}$

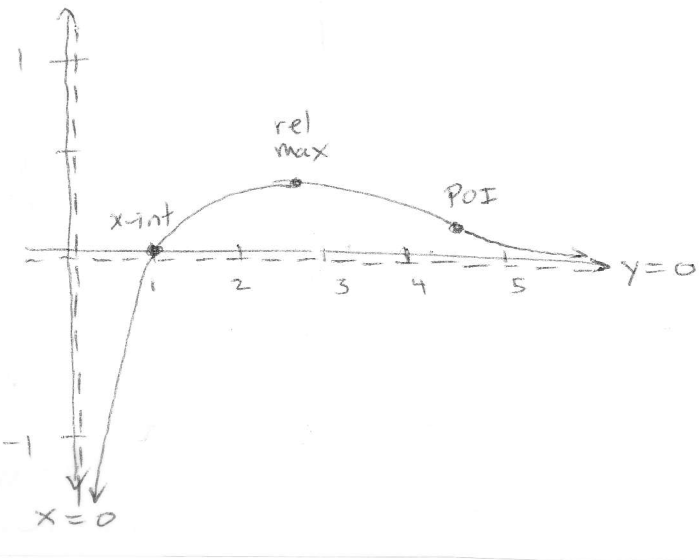
horizontal asymptote: $\lim_{x \to \infty} \frac{\ln x}{x} = 0$ but we can't prove it until Math 251 $\left\{ \text{chart: } f(1{,}000{,}000) \approx 1.4 \times 10^{-4} \right\}$

increasing $(0, e]$

decreasing $[e, \infty)$

concave $(0, e^{3/2})$

concave $(e^{3/2}, \infty)$



$(e, \frac{1}{e}) \approx (2.7, .4)$ rel. max.

$(e^{3/2}, \frac{3}{2}e^{-3/2}) \approx (4.5, .3)$ POI

$(1, 0)$ x int.

domain $(0, \infty)$

M250 [redacted]

9

⑪ Find the exact value of $6^{\ln 10/\ln 6}$ without a calculator.

Call it x: $6^{\ln 10/\ln 6} = x$

Take ln both sides: $\ln 6^{\ln 10/\ln 6} = \ln x$

log properties: $\frac{\ln 10}{\ln 6} \cdot \ln 6 = \ln x$

divide out common factor $\frac{\ln 6}{\ln 6} = 1$:

$$\ln 6 = \ln x$$

exponentiate both sides: $\boxed{6 = x}$

Math 250

⑫ Richter scale for earthquakes

$$\log E = 4.4 + 1.5 M$$

E = released energy in joules
M = magnitude

Suppose two earthquakes differ by 1 on the Richter scale. Find the ratio of the released energy of the larger earthquake to that of the smaller earthquake.

larger E_1 = energy
M_1 = magnitude

smaller E_2 = energy
M_2 = magnitude

magnitudes differ by 1 : $M_1 - M_2 = 1$
$M_1 = M_2 + 1$

find ratio $\frac{E_1}{E_2}$.

Since Richter is a log scale, take logs of this ratio.

$\log\left(\frac{E_1}{E_2}\right) = \log E_1 - \log E_2$ log property

$= (4.4 + 1.5M_1) - (4.4 + 1.5M_2)$ subst defn.

$= 4.4 + 1.5M_1 - 4.4 - 1.5M_2$

$= 1.5 M_1 - 1.5 M_2$

$= 1.5(M_2+1) - 1.5 M_2$ subst relationship

$= 1.5 M_2 + 1.5 - 1.5 M_2$

$= 1.5$ Aha! M_2 disappeared!

$\log\left(\frac{E_1}{E_2}\right) = 1.5$ beginning = end

$10^{1.5} = \frac{E_1}{E_2}$ equivalent exponential

$\boxed{\frac{E_1}{E_2} \approx 31.6}$ About 32 times as much energy is released.

Math 250

A limit involving e :

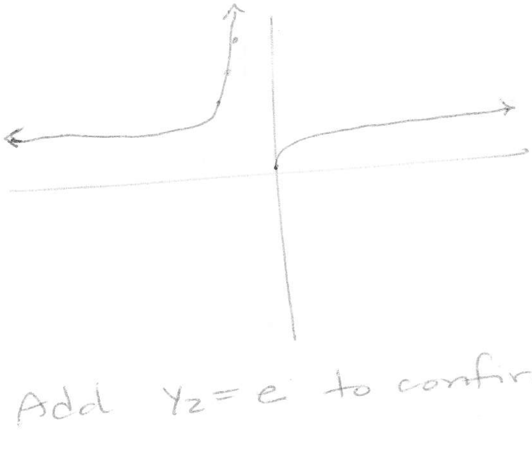
$$\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x = e.$$

GRAPH $Y_1 = (1 + 1/x)$^x on GC.

TBLSET — ASK

x	$Y_1(x)$
1	2
10	2.5937424601
100	2.70481382942
1000	2.71692393224
10000	2.71814592683
100000	2.71826823717
1000000	2.71828046932
10000000	2.71828169255

$\boxed{\text{2nd}}\boxed{\div} = e \Rightarrow$ 2.718281828459045



use dot mode to see that it's discontinuous

Add $Y_2 = e$ to confirm asymptote differently.

continued next page

250
5.5

A Limit involving e. (Proof from Appendix A).

$$y = \lim_{x \to \infty}\left(1+\frac{1}{x}\right)^{x}$$

$$y = \lim_{x \to \infty}\left(\frac{x+1}{x}\right)^{x}$$

$$\lim_{x \to x_0} f(x) = f\left(\lim_{x \to x_0} x\right)$$

continuous functions

$$\ln y = \ln\left(\lim_{x \to \infty}\left(\frac{x+1}{x}\right)^{x}\right)$$

$$\ln y = = \lim_{x \to \infty}\left[\ln\left(\frac{x+1}{x}\right)^{x}\right]$$

$$\ln y = \lim_{x \to \infty}\left[x \cdot \ln\left(\frac{x+1}{x}\right)\right]$$

$$\ln y = \lim_{x \to \infty}\left[x \cdot \ln\left(1+\frac{1}{x}\right)\right]$$

$$\ln y = \lim_{x \to \infty}\left[\frac{\ln\left(1+\frac{1}{x}\right)}{(1/x)}\right]$$

$t = \frac{1}{x}$ as $x \to +\infty$, $t \to 0^{+}$

$$\ln y = \lim_{t \to 0^{+}}\left[\frac{\ln(1+t)}{t}\right]$$

$$\ln y = \lim_{t \to 0^{+}}\left[\frac{\ln(1+t) - 0}{t}\right]$$

$$\ln y = \lim_{t \to 0^{+}}\left[\frac{\ln(1+t) - \ln(1)}{t}\right]$$

$$\ln y = \left[\frac{d}{dx}\ln x\right]\Big|_{x=1}$$

$$\ln y = \frac{1}{x} \text{ at } x = 1$$

$$\ln y = 1$$

$$e^{1} = y$$

$\therefore \lim = e.$

Extra Exploration on Log Differentiation

Remember that $\log_b(x)$ undefined for $x \le 0$, any base $b>0$, $b \neq 1$.

So when we take ln of both sides of an equation,

a) Is that always defined for the same domain as the original?

[No.]

b) When we expand using log properties do we get domains different from the original?

[YES.]

c). So why is log differentiation valid?

[BECAUSE WE CAN TAKE THE DOMAIN CASE-BY-CASE FOR DIFFERENT SEGMENTS OF THE DOMAIN]

Example: $f(x) = (x-1)(x+2)$

We know $f(x) = x^2 + x - 2$

with $f'(x) = 2x+1$ for any $x \in (-\infty, \infty)$.

But what if we find this derivative using log diff?

$y = (x-1)(x+2)$ domain: all $\mathbb{R}$

$\ln y = \ln[(x-1)(x+2)]$ domain: $(-\infty,-2) \cup (1,\infty)$

$\ln y = \ln(x-1) + \ln(x+2)$ domain: $(1,\infty)$

$$\frac{1}{y}\cdot\frac{dy}{dx} = \frac{1}{x-1} + \frac{1}{x+2}$$

$$\frac{dy}{dx} = y\left[\frac{1}{x-1} + \frac{1}{x+2}\right]$$

$$\frac{dy}{dx} = (x-1)(x+2)\left[\frac{1}{x-1} + \frac{1}{x+2}\right]$$

$$\frac{dy}{dx} = x+2 + x-1$$

$$\frac{dy}{dx} = 2x+1$$

technically this result is valid only $x>1$!

To explain this, we break the domain in 4 cases and play sign games.

Extra Exploration con't

When a function has values which are negative, we cannot take ln of negative y-values, since the domain of ln is positive numbers only.

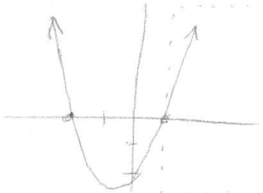
However we can consider the negative to be a constant multiple: $y = -f(x)$

means

$$\frac{dy}{dx} = -\frac{df(x)}{dx}$$

and write a piecewise function

Example $y = (x-1)(x+2)$ has graph



meaning that

$$y = \begin{cases} (x-1)(x+2) & \text{when } x < -2 \text{ or } x > 1 \\ -(x-1)(x+2) & \text{when } -2 \le x \le 1 \end{cases}$$

We can apply ln differentiation to $(x-1)(x+2)$ for $(-\infty,-2) \cup (1,\infty)$

and obseve that for $(-2,1)$ $\frac{dy}{dx} = -\frac{d}{dx}\left[-(x-1)(x+2)\right]$

CASE #1: $x > 1$ (No problems)

$$y = (x-1)(x+2)$$

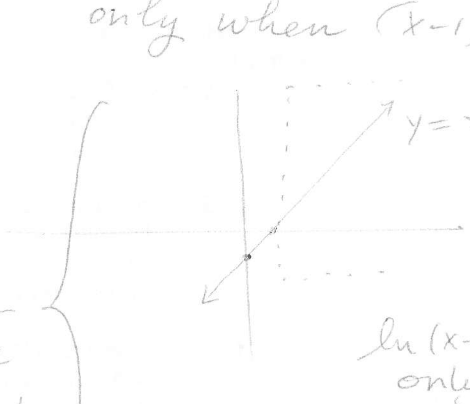
$$\ln y = \ln(x-1) + \ln(x+2)$$ only when $(x-1)(x+2) > 0$

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x-1} + \frac{1}{x+2}$$

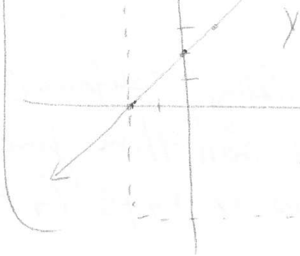
$$\frac{dy}{dx} = (x-1)(x+2)\left[\frac{x+2+x-1}{(x-1)(x+2)}\right]$$

$$\frac{dy}{dx} = 2x+1$$ on $(1,\infty)$

Together these are defined only on $(1,\infty)$



ln(x-1) defined only if x>1



ln(x+2) defined only if x>-2

CASE 2: $x < -2$

Extra Exploration (cont)

$$y = (x-1)(x+2)$$

$$y = (1-x)(-2-x)$$

For $x < -2$

need that

$$-(1-x)(2+x) = (x-1)(x+2)$$

$$(1-x)(-2-x)$$

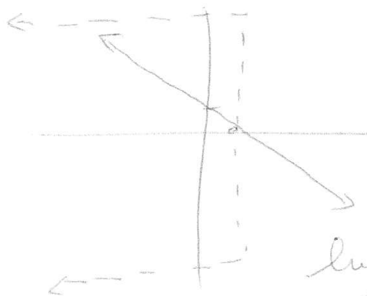
$$\ln y = \ln(1-x) + \ln(-2-x)$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{-1}{1-x} + \frac{-1}{-2-x}$$

$$\frac{dy}{dx} = (1-x)(-2-x)\left[\frac{2+x+x-1}{(1-x)(-2-x)}\right]$$

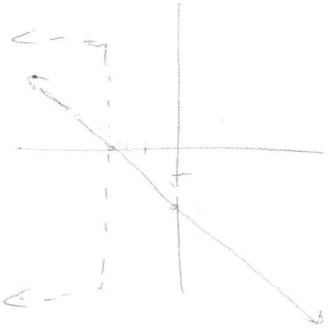
$$\frac{dy}{dx} = 2x+1 \text{ on } (-\infty,-2)$$

Together these are defined for $(-\infty,-2)$



$y = -x+1$

$\ln(-x+1)$ defined only for $x<1$



$y = -x-2$

$\ln(-x-2)$ defined only for $x<-2$

CASE 3: $-2 < x < 1$

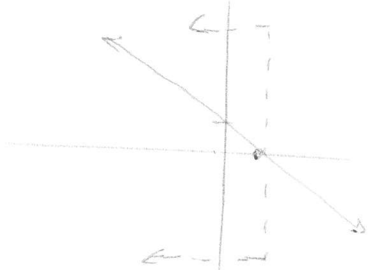
But for $(-2,1)$ we need to use

$$(-x+1)(x+2) = -(x-1)(x+2)$$

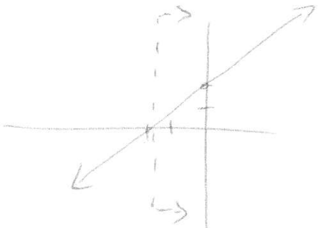
We observe that

$$-(-x+1)(x+2) = (x-1)(x+2)$$

But we can use ln diff only on $(-x+1)(x+2)$ noticing that this is everywhere negative, meaning that we use ln diff on $(-x+1)(x+2)$ then multiply result by -1, a constant multiple.



$y = -x+1$



$y = x+2$

$$y_2 = (-x+1)(x+2) = -y_1$$

$$\ln y_2 = \ln(-x+1) + \ln(x+2)$$

$$\frac{1}{y_2}\frac{dy_2}{dx} = \frac{-1}{-x+1} + \frac{1}{x+2}$$

Math 250 Extra Exploration cont.

$$\frac{dy_2}{dx} = y_2\left[\frac{-x-2+-x+1}{(-x+1)(x+2)}\right]$$

$$\frac{dy_2}{dx} = (-x+1)(x+2)\left[\frac{-2x-1}{(-x+1)(x+2)}\right]$$

$$\frac{dy_2}{dx} = -2x-1$$

So that $\frac{dy}{dx} = -(-2x-1) = 2x+1$ on $(-2,1)$ also

Fortunately,

$$y'(x) = \begin{cases} 2x+1 & x<-2 \\ 2x+1 & -2<x<1 \\ 2x+1 & x>1 \end{cases}$$

only the endpoints $x=-2$ and $x=1$ are still unclear

CASE 4: $x=-2$ or $x=1$

$$\lim_{x\to -2} \frac{f(x)-f(-2)}{x+2} = 2x+1$$

$$\lim_{x\to 1} \frac{f(x)-f(1)}{x-1} = 2x+1$$

$\therefore\ f'(x) = 2x+1 \ \forall x.$

Q: Can log diff be used on $y = -\sqrt{x^2(x-1)(x+2)}$ if $x>0$?

A: Yes, if we notice that $y=-y_1$ where $y_1=\sqrt{x^2(x-1)(x+2)}$ and notice that neither y nor y_1 is defined $(-2,1)$. so it's not differentiable at $x=-2$ or $x=1$.

while log diff cannot be applied to y, it can be applied to y_1 to get $\frac{dy_1}{dx}$.

Then we multiply by -1 to get $\frac{dy}{dx} = -\frac{dy_1}{dx}$.